

GRAVITATIONAL RADIATION FROM ROTATING GRAVITATIONAL COLLAPSE

RICHARD F. STARK
Institute for Theoretical Physics
University of California, Santa Barbara, CA 93106, U.S.A.

ABSTRACT

We find the efficiency of gravitational wave emission from axisymmetric rotating collapse to a black hole to be very low: $\Delta E / Mc^2 < 7 \times 10^{-4}$. The main waveform shape is well defined and nearly independent of the details of the collapse. Such a signature will allow pattern recognition techniques to be used when searching experimental data. These results (which can be scaled in mass) have been obtained using a fully general relativistic computer code that evolves rotating axisymmetric configurations and directly computes their gravitational radiation emission.

The results summarized here have come from a tested, fully general relativistic computer code able to evolve axisymmetric rotating matter configurations. This code numerically solves the complete coupled Einstein and hydrodynamic equations for axisymmetric systems. A more detailed description of the results, the code (and its testing), and the formalism used can be found in Stark and Piran (1985, 1986, and 1987). (A possible extension to non-axisymmetric collapse is given in Stark 1988.)

The axisymmetric rotating gravitational collapse we have studied is that of an initially pressure-deficient, 'rigidly rotating' polytrope with an adiabatic equation of state (Stark and Piran (1985, 1986, and 1987)). We first set up a TOV spherically symmetric non-rotating polytrope. For the cases studied here, we have used an initial stellar radius of a $6GM/c^2$ and a fixed adiabatic index $\Gamma = 2$. We then reduce the pressure and thermal energy to a fraction f_p of their equilibrium values (with $f_p = 0.01$ or 0.4) and simultaneously add a 'rigid body' azimuthal rotation to the star. The rotation is measured by the dimensionless angular momentum parameter $a = J / \frac{GM^2}{c}$ where J is the total angular momentum of the star. The collapse and resulting gravitational wave emission are studied for a range of angular momenta $0 \leq a < 1.5$. With our chosen equation of state, all quantities scale in an elementary fashion with M the mass of the star. Our initial conditions correspond to an initial radius of $8.8 \times 10^5 M/M_\odot \text{cm}$ and a central density of $1.9 \times 10^{15} (M/M_\odot)^{-2} \text{gcm}^{-3}$ (corresponding to a somewhat unrealistic, compact, relativistic stellar core for stellar masses). (For space-based experiments one can simply scale to higher M , thus approximating supermassive or population III collapse.)

(For illustrations see the figures in Stark and Piran 1986). For $a = 0.40$ (and $f_p = 0.4$), the collapse takes place almost spherically symmetrically with the meridional velocity vector remaining closely radial, little flattening of the star, and without any shocks forming. The central density increases by a factor of ~ 15 and after a time of $\sim 35M$ the star 'freezes' as the lapse (which relates proper and coordinate times) drops exponentially due to black hole formation. For the collapse of a more rapidly rotating star with $a = 0.75$ (and $f_p = 0.4$), the rotational effects

become noticeable with the star soon becoming highly flattened into the equatorial plane. After a time of $\sim 30M$, the star bounces vertically away from the equator, but still continues to collapse radially inwards until freezing as a black hole forms, and the lapse collapses. For $a > a_{crit}$ (with $a_{crit} = 0.85 \pm 0.05$ for $f_p = 0.4$), the effects of rotation become dominant, no black hole forms, and the star ends up oscillating about a flattened rotating equilibrium configuration (see Stark and Piran 1986). (The general behavior for $f_p = 0.01$ is similar to that described, except that the star becomes rather more flattened and (for $a < a_{crit}$) collapses slightly faster).

Fig. 1 shows results obtained for the even and odd transverse traceless waveforms h_+, h_\times (as defined in Stark and Piran 1987) from the collapse to black holes of stars with various $a < a_{crit}$ (and for $f_p = 0.01$). The waveforms shown have been monitored at the outer edge of the grid ($r \sim 50M$). The even and odd modes have the expected $\sin^2\theta$ and $\cos\theta\sin^2\theta$ angular dependences respectively, and these as well as the r^{-1} radial fall-off have been factored out. The waveform is well established by fairly small radii ($r \sim 25M$). The waveform consists of a broad peak at a retarded time ~ 0 due to the initial flattening of the star from its initial spherical configuration. (This part contributes negligibly to the total energy of the gravitational wave emission.) The main emission appears at a retarded time of $\sim 25M$, corresponding to when the collapsing star is only $\sim 3M$ in size. The main emission lasts for $\sim 10M$ and is followed by a decaying oscillatory tail. The most noticeable feature of these waveforms is the insensitivity of the waveform shape to the value of a . Over the range $0 < a < a_{crit}$ for black hole collapse, the shape remains closely unchanged, the waveform differing for each collapse only in its amplitude. (This is the case for $f_p = 0.4$ also.)

The maximum h_+ amplitude, $|h_+|_{max}$, from the collapse to a black hole, both for $f_p = 0.01$ and 0.4 , scales very closely as a^2 and finally levels off to a maximum value as $a \rightarrow a_{crit}$. Our results are closely approximated by:

$$(r/M)|h_+|_{max} = \min\{0.1a^2, A_{max}\} \quad (0 < a < a_{crit})$$

with $A_{max} = 0.06$, $a_{crit} = 1.2 \pm 0.2$ for $f_p = 0.01$; and $A_{max} = 0.025$, $a_{crit} = 0.8 \pm 0.05$ for $f_p = 0.4$. (As would be expected for axisymmetric collapse $|h_\times| < 0.2|h_+|$.)

The dependence of the total gravitational wave energy emitted, ΔE , on the angular momentum of the collapse (to a black hole) is shown in Fig. 2 for both $f_p = 0.01$ and 0.4 . The most noticeable feature is the very low efficiency $\frac{\Delta E}{Mc^2}$. In all cases the efficiency is $< 7 \times 10^{-4}$. The emitted energy ΔE scales as a^4 (as shown by the solid line fit in Fig. 2) with a coefficient of 1.4×10^{-3} independent of f_p , and ΔE levels off to a maximum value ϵ_{max} (which depends on f_p , as shown by the dotted lines) as $a \rightarrow a_{crit}$. Our results closely follow the form:

$$\Delta E/Mc^2 = \min\{1.4 \times 10^{-3}a^4, \epsilon_{max}\} \quad (0 < a < a_{crit})$$

with $\epsilon_{max} = 6 \times 10^{-4}$ for $f_p = 0.01$; and $\epsilon_{max} = 1 \times 10^{-4}$ for $f_p = 0.4$. In all cases $(\Delta E/Mc^2) > 10(\Delta E/Mc^2)_\times$.

Corresponding to the insensitivity of the waveform shape, the shape of the energy spectra, from these collapses to a black hole is similarly insensitive. The spectra have the form of a simple peaked curve, although for values of a near a_{crit} , a low frequency component due to the initial flattening of the star appears (see Stark and Piran 1986 for detailed figures). The energy spectra peak at frequencies of $0.035 < \nu < 0.08 \left(\frac{GM}{c^3}\right)^{-1} \text{ Hz}$ (*i.e.*, $7 < \nu < 16 (M/M_\odot)^{-1} \text{ kHz}$) corresponds to wavelengths of between 12 and 28 $\left(\frac{GM}{c^2}\right)$. The peak spectral energy $F(\nu_{max})$ scales as a^4 , and levels off to a maximum value as $a \rightarrow a_{crit}$. Our numerical results closely follow the relations:

$$F(\nu_{max}) \text{ erg cm}^{-2} \text{ Hz}^{-1} = \min\{60a^4, F_{max}\} (M/M_\odot)^2 (r/10\text{kpc})^{-2}$$

with $F_{max} = 22$ for $f_p = 0.01$; and $F_{max} = 4$ for $f_p = 0.4$.

Our waveforms are very similar in shape to those of several previous perturbation studies (see Stark and Piran 1986 for details). This broad agreement in waveform shapes suggests the importance of black hole quasi-normal mode excitation in the gravitational wave emission process. The collapsing star produces most gravitational wave emission when it is $\sim 3M$ in size, by which time the exterior metric is already quite similar to that of a black hole. The star thus acts as a source to excite the black hole quasi-normal modes. (The star being smaller than the peak of the black hole potential determining the normal modes.) The waveform shape reflects the properties of the black hole formed, whereas the amplitude to which the modes are excited depend on the details of the collapse. The observed insensitivity of the shape of the waveform to the rotation parameter a , corresponds to the weak dependence on a of the $m = 0$ axisymmetric modes of the Kerr black hole (Detweiler 1980). This picture is confirmed by the very good fit we obtain to the waveform from a linear combination of the two lowest modes (see Stark and Piran 1986).

We may summarize our results as follows. The most optimistic case (*i.e.*, the one producing the largest gravitational radiation) is for $a \sim a_{crit}$ and a small initial pressure ($f_p = 0.01$). For this case, we expect the maximum observable strain $|\Delta l/l|_{max}$ to be:

$$|\Delta l/l|_{max} = \begin{cases} |h_+|_{max} = 4.0 \times 10^{-19} (M/1.4M_\odot) (r/10\text{kpc})^{-1} \\ \sqrt{G/c^3 F(\nu_{max})} = 3.6 \times 10^{-19} (M/1.4M_\odot) (r/10\text{kpc})^{-1} \end{cases}$$

where the two measures of the strain correspond to nonresonant (interferometer) and resonant (bar) experiments respectively. The maximum spectral energy, peak frequency, and full-width, half-maximum frequency are:

$$F(\nu_{max}) = 45 (M/1.4M_\odot)^2 (r/10\text{kpc})^{-2} \text{ erg cm}^{-2} \text{ Hz}^{-1}$$

$$\nu_{max} = 10 (M/1.4M_\odot)^{-1} \text{ kHz}; \Delta\nu_{1/2} = 4 (M/1.4M_\odot)^{-1} \text{ kHz}$$

For low a collapses ($a < 0.8$), the expected maximum strain and maximum spectral energy are:

$$|\Delta l/l|_{max} = 6.7 \times 10^{-19} a^2 (M/1.4M_{\odot})(r/10\text{kpc})^{-1} \quad (\text{both cases})$$

$$F(v_{max}) = 120 a^4 (M/1.4M_{\odot})^2 (r/10\text{kpc})^{-2} \text{erg cm}^{-2} \text{Hz}^{-1}$$

with v_{max} and $\Delta v_{1/2}$ as above.

An important feature of the main black hole emission is that the general waveform *shape* is well defined and nearly independent of the details of the collapse. This provides a signature for the black hole collapse and will allow pattern recognition techniques to be applied to broadband experimental gravitational wave data, thus allowing greater sensitivities to be reached for these events. Further experimental implications for ground-based bar detectors of stellar collapse can be found in Stark and Piran 1986.

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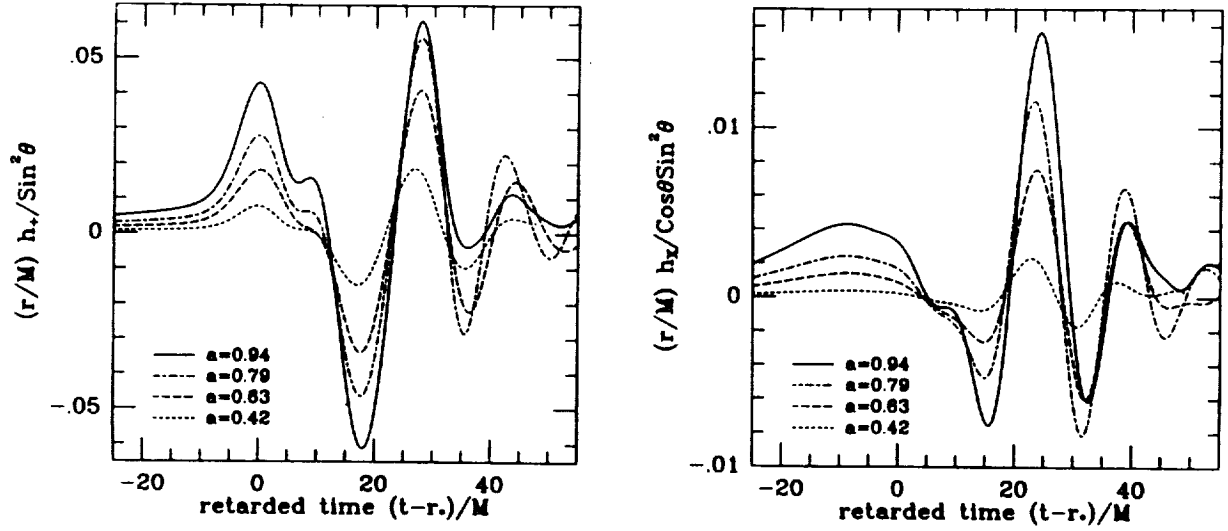


FIG. 1 — h_+ (left) and h_{\times} (right) waveforms vs. retarded time for various a ($f_p = 0.01$) black hole collapses.

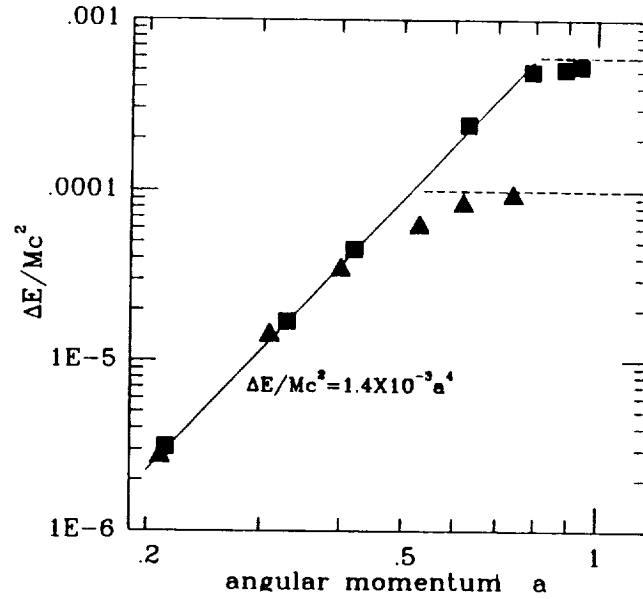


FIG. 2 — Total Gravitational wave energy (ADM) emitted vs. a for $f_p = 0.01$ (squares) and $f_p = 0.4$ (triangles) black hole collapses. Solid line fits the a^4 scaling; dashed lines mark ϵ_{\max} .

DISCUSSION

ARMSTRONG: How would you expect the shape of gravitational temporal waveforms to change for non-axisymmetric collapse? Gravity wave efficiencies?

STARK: The waveforms from rotating non-axisymmetric black hole collapse can be expected to consist of a quasi-periodic phase (due to the rotation of the bar) followed by emission from the actual black hole formation and a decaying tail. The main black hole emission can be expected to show similar normal mode characteristics as we have seen in the axisymmetric collapse. Thus we may expect this part of the emission to have a shape similar to that for the axisymmetric collapse, except possible at high rotations where the non-axisymmetric normal mode frequencies become appreciably split. As far as the efficiency is concerned, perturbation studies indicate that it may be at least an order of magnitude larger than for axisymmetric collapse (for an appreciable level of non-axisymmetry). We will not know for sure until numerical simulations are made. (See e.g., Stark, 1988 for work in progress).

HELLINGS: Is it true that changing "a" changes the amplitude without changing the period while changing "m" changes both in a proportional way?

STARK: The waveform amplitude from collapse to a black hole scales as a^4 for low a (where $a = J/GM^2/c$; J the total angular momentum; M the mass of the star) and then levels off before a reaches a_{crit} (the rotation above which black hole formation no longer occurs). The amplitude also scales proportionally with M . The waveform period (or more precisely, the scale of retarded time for the waveform) of the main emission from black hole formation is insensitive to a (due to the insensitivity of the axisymmetric Kerr black hole normal modes frequencies to a). The period scales proportionally with M .

SHAPIRO: Given the complexity of the system you're modelling, and the (necessarily) complicated code needed to solve the coupled equations used to represent the system's evolution, what checks have been made to try to insure that the results are correct?

STARK: It is clearly very important that a code of this kind be tested as fully as possible. We have performed an extensive series of tests which can be broadly classified into general stability tests, conservation tests, hydrodynamic tests and comparison tests against known approximate perturbation solutions. (The derivation of the equations we solve has also been checked using symbolic computer algebra). A detailed description of these tests can be found in Stark & Piran, 1987. The tests performed include: (i) Evolution of the vacuum Schwarzschild exterior ($r > 2M$) for many gravitational times; (ii) Propagation of gravitational waves (for both polarizations) along inward and outward characteristics with negligible reflection at the outer boundary; evolution to flat spacetime for low amplitude waves, and black hole formation for high initial wave amplitudes; (iii) Stable evolution of initial data consisting of flat spacetime + a small amplitude 'random data' superimposed; (iv) Evolution of stable extreme relativistic polytropes for many free fall timescales; including checking the excitation of the lowest radial mode oscillations for adiabatic indices $\Gamma > \Gamma_{\text{crit}}$, as well as the instability to collapse for $\Gamma < \Gamma_{\text{crit}}$ (with $\Gamma_{\text{crit}} = 4/3$ + general relativistic corrections); (v) Conservation of the ADM mass and the total angular momentum; (vi) Conservation of the specific angular momentum spectrum; (vii) Propagation of generalized linearized Teukolsky waves for both polarization modes; (viii) Comparison of the gravitational wave emission from the infall of a

spheroidal dust shell onto a Schwarzschild black hole with known perturbation results; and (ix) Comparison of the gravitational wave emission and hydrodynamics for uniformly rotating homogeneous spheroidal collapse with known Newtonian + quadrupole formalism results. It is also worth remarking that the waveform obtained from the full scale calculation of the axisymmetric collapse to a black hole shows a remarkable agreement with the waveform from an approximate perturbation study (Nakamura, Oohara & Kojima, Prog. Theor. Phys. Suppl., No. 90, 1987) of a rotating dust ring falling in from infinity onto a Schwarzschild black hole with a specific angular momentum comparable to that of the collapsing star. The collapsing dust ring mimics the flattening in the equatorial plane of the collapsing star. One finds very near agreement in the waveform shapes, and, on scaling up the perturbation result (beyond its actual range of validity) the amplitudes as well. (Both efficiencies then also agree at a few $\times 10^{-4}$).

BENDER: How likely does it seem that some instability such as a bar instability would occur during collapse? What mechanisms might determine the initial axial asymmetry?

STARK: A bar mode instability is very likely for sufficiently high rotation. The amount of non-axisymmetry depends on the exact details of the microphysics and initial data. At present these are not known sufficiently well. Work in progress on non-axisymmetric collapse to a black hole (Stark, 1988) will investigate how the gravitational emission varies for a range of non-axisymmetry, but will not attempt to answer which level of non-axisymmetry is astrophysically appropriate.

SCHUTZ: I will just comment on the question of the generation of non-axisymmetry in collapse. Because it relies on an instability that grows exponentially, only a very small initial perturbation is needed. The instability is the 'bar mode' instability that has been discussed as well in the context of formation of bars and spirals in galaxies. Basically, if angular momentum is conserved in the collapse, it becomes energetically favorable for the system to deform into a tumbling bar. The hard question for numerical calculations to answer, and for which careful attention to the microphysics may be necessary, is how much angular momentum transport there may be as the object collapses.